Explicit Schilder Type Theorem for Super-Brownian Motions

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Outline

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Abstract

• Like ordinary Brownian motion, super-Brownian motion, a central object in the theory of superprocesses, is a universal object arising in a variety of different settings.

• Schilder type theorem and Cramér type one are two of major topics for the large deviation theory.

• A Schilder type, which is also a Cramér type, sample large deviation for super-Brownian motions with a good rate function represented by a variation formula was established around 1993/1994; and since then there have been several efforts making very valuable contributions to give an affirmative answer to the question whether this sample large deviation holds with an explicit good rate function.

• Thanks to previous results on the mentioned question, and the Brownian snake, we establish such a kind of large deviations for all nonzero initial measures.

Outline

The State Space of SBM

- **One state space:** $\mathcal{M}_r\left(\mathbb{R}^d\right)$. For fixed $r > d/2$, write

$$\phi_r(x) = \left(1 + |x|^2\right)^{-r}, \ \forall x \in \mathbb{R}^d.$$

Denote by

$$\langle \mu, f \rangle = \mu(f) = \int_{\mathbb{R}^d} f(x) \ \mu(dx)$$

the integral of a function f against a measure μ on $\mathbb{R}^d$ if it exists. Let $\mathcal{M}_r\left(\mathbb{R}^d\right)$ be the set of all Radon measures μ on $\mathbb{R}^d$ with $\langle \mu, \phi_r \rangle < \infty$, and endow it with the following τ_r topology:

$$\mu_n \Longrightarrow \mu \text{ iff } \lim_{n\to\infty} \langle \mu_n, f \rangle = \langle \mu, f \rangle, \ \forall f \in C_c\left(\mathbb{R}^d\right) \cup \{\phi_r\}.$$

- **Another state space:** $\mathcal{M}_F\left(\mathbb{R}^d\right)$. Let $\mathcal{M}_F\left(\mathbb{R}^d\right)$ be the space of all finite measures on $\mathbb{R}^d$. Endow it with the weak convergence topology.

Two Trajectory Spaces

- **Two continuous trajectory spaces:** Ω_μ **and** $\Omega_{\mu,F}$

$$\Omega_\mu = \left\{\omega = (\omega_t)_{0\le t\le 1} : \ \omega \in C\left([0,1], \mathcal{M}_r\left(\mathbb{R}^d\right)\right), \ \omega_0 = \mu\right\}$$
$$\text{for any } \mu \in \mathcal{M}_r\left(\mathbb{R}^d\right).$$

$$\Omega_{\mu,F} = \left\{\omega = (\omega_t)_{0\le t\le 1} : \ \omega \in C\left([0,1], \mathcal{M}_F\left(\mathbb{R}^d\right)\right), \ \omega_0 = \mu\right\}$$
$$\text{for any } \mu \in \mathcal{M}_F\left(\mathbb{R}^d\right).$$

Introduce SBM

Let Δ be the Laplace operator on $\mathbb{R}^d$ and $\mathcal{B}_b\left(\mathbb{R}^d\right)$ the set of bounded measurable functions on $\mathbb{R}^d$. Fix $\sigma, \rho \in (0, \infty)$. Denote by $\{S_t^\sigma\}_{t\geq 0}$ the semigroup with the generator $\sigma\Delta$.

• Then SBM $X = (X_t)_{t\geq 0}$ is the unique diffusion on $\mathcal{M}_r\left(\mathbb{R}^d\right)$ such that for any $\mu \in \mathcal{M}_r\left(\mathbb{R}^d\right)$, $0 \leq f \in \mathcal{B}_b\left(\mathbb{R}^d\right)$,

$$\mathbb{E}\left[\exp\{-\langle X_t, f\rangle\} \mid X_0 = \mu\right] = \exp\left\{-\left\langle \mu, u_t^{\sigma,\rho} f\right\rangle\right\}, \ t \geq 0.$$

Where $u_t^{\sigma,\rho} f$ is the unique nonnegative solution to the equation

$$u_t = S_t^\sigma f - \frac{\rho}{2}\int_0^t S_{t-s}^\sigma\left[(u_s)^2\right] ds, \ t \in \mathbb{R}_+.$$

Note X corresponds to the branching mechanism

$$z \in \mathbb{R}_+ \to \frac{\rho}{2} z^2 \in \mathbb{R}_+.$$

Introduce SBM

- Let $P_{\mu}^{\sigma,\rho}$ be the distribution of SBM $(X_t)_{0\leq t\leq 1}$ $(X_0=\mu)$ on Ω_{μ}.
- Note for any $\mu\in\mathcal{M}_F\left(\mathbb{R}^d\right)$, $P_{\mu}^{\sigma,\rho}$ is a probability on $\Omega_{\mu,F}$.

Introduce SBM

• SBM was introduced independently by S. Watanabe (1968) and D. A. Dawson (1975). The name

"Super-Brownian motion (Superprocess)"

was coined by E. B. Dynkin in the late 1980s.

• SBM is the most fundamental branching measure-valued diffusion process (superprocess).

• Like ordinary Brownian motion, SBM is a universal object which arises in models from combinatorics (lattice tree and algebraic series), statistical mechanics (critical percolation), interacting particle systems (voter model and contact process), population theory and mathematical biology, and nonlinear partial differential equations. Refer to J. F. Le Gall (1999), G. Slade (2002) et al.

Outline

The LDP for SBM

We study the LDP for $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ on Ω_μ as $\epsilon \downarrow 0$ for $\mu \neq 0$. This is a Freidlin-Wentzell type LDP ([K. Fleischmann, J. Gärtner and I. Kaj (1996) *Can. J. Math.*] called it a Schilder type LDP by comparing the form of its possible rate function with that of Schilder theorem for Brownian motions). On the other hand, since

$$P_\mu^{\sigma,\epsilon\rho}[(X_t)_{0\leq t\leq 1} \in \cdot\ ] = P_{\mu/\epsilon}^{\sigma,\rho}[(\epsilon X_t)_{0\leq t\leq 1} \in \cdot\ ],\ \forall \epsilon \in (0,\infty),$$

and for any natural number n, $P_{n\mu}^{\sigma,\rho}\left[\left(\frac{1}{n}X_t\right)_{0\leq t\leq 1} \in \cdot\ \right]$ is the law of the empirical mean of n independent SBMs distributed as $P_\mu^{\sigma,\rho}$ due to the branching property; the mentioned LDP is an infinite-dimensional version of the well-known classical Cramér theorem with continuous parameters ([A. Schied (1996) *PTRF*]).

The LDP for SBM

• *The LDP for* $\left\{P_{\mu}^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ *is both Freidlin-Wentzell type and Cramér type at the same time, which is a fascinating feature.*

• When μ is the Lebesgue measure on $\mathbb{R}^d$, LDP for $\left\{P_{\mu}^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ as $\epsilon \downarrow 0$ is equivalent to an occupation time LDP for SBMs relating to mass-time-space scale transformations. For LDPs concerning the unscaled ergodic limits of occupation time of SBMs, refer to

[J. D. Deuschel, J. Rosen (1998) *Ann. Probab.*],
[T. Y. Lee, B. Remillard (1995) *Ann. Probab.*],
[T. Y. Lee (2001) *Ann. Probab.*].

For other large deviation results for some super-Brownian processes, see [L. Serlet (2009) *Stoc. Proc. Appl.*] and references therein.

The LDP for SBM

• Note the first statement of Cramér theorem (on $\mathbb{R}^1$) was due to [H. Cramér (1938)]; and [M. D. Donsker, S. R. S. Varadhan (1976) *Comm. Pure. Appl. Math*] extended firstly Cramér theorem to separable Banach spaces.

• While classical Schilder theorem was derived firstly by [M. Schilder (1966) *Trans. AMS.*]; and then based on Fernique's inequality, it was generalized to abstract Wiener space by [J. D. Deuschel, D. W. Stroock (1989)]. A non-topological form of Schilder theorem for centered Gaussian processes based on the isoperimetric inequality was established by [G. Ben Arous, M. Ledoux (1993)].

Outline

The rate function and conjecture

• Furnish the Schwartz space $\mathcal{D} := C_c^{\infty}\left(\mathbb{R}^d\right)$, the set of all smooth functions on $\mathbb{R}^d$ with compact supports, with its inductive topology via the subspaces

$$\mathcal{D}_K = \{\phi \in \mathcal{D} \mid \phi(x) = 0, \ \forall x \notin K\},$$

here sets K are compact in $\mathbb{R}^d$. Write $\mathcal{D}^*$ for the dual space of $\mathcal{D}$, namely the space of all Schwartz distributions on $\mathbb{R}^d$. Clearly, $\mathcal{M}_r\left(\mathbb{R}^d\right) \subset \mathcal{D}^*$.

• For any $\nu \in \mathcal{M}_r\left(\mathbb{R}^d\right)$, define $\Delta^*\nu \in \mathcal{D}^*$, by

$$\langle \Delta^*\nu, f\rangle = \langle \nu, \Delta f\rangle, \ \forall f \in \mathcal{D}.$$

The rate function and conjecture

- We call a map

$$t \in [0,1] \to \theta_t \in \mathcal{D}^*$$

absolutely continuous if for any compact set K in $\mathbb{R}^d$, there are a neighborhood V_K of 0 in $\mathcal{D}_K$ and an absolutely continuous real valued function h_K on $[0,1]$ satisfying

$$|\langle \theta_t, f\rangle - \langle \theta_s, f\rangle| \leq |h_K(t) - h_K(s)|\,, \ \forall s,t \in [0,1], \ \forall f \in V_K.$$

For such a map, the derivatives $\dot{\theta}_t \in \mathcal{D}^*$ exist in the Schwartz distributional sense for almost all $t \in [0,1]$ with respect to the Lebesgue measure. See

[D. A. Dawson, J. Gärtner (1987) *Stochastics*].

The rate function and conjecture

- Let H_μ^σ be the space of all absolutely continuous (in time t in sense of Schwartz distributions) paths $\omega = (\omega_t)_{0\le t\le 1} \in \Omega_\mu$ with derivatives $\dot{\omega}_t$ such that for almost every t with respect to the Lebesgue measure, the Schwartz distribution $\dot{\omega}_t - \sigma\Delta^*\omega_t$ is absolutely continuous with respect to ω_t, and

$$\int_0^1 \left\langle \omega_t, \left| \frac{d(\dot{\omega}_t - \sigma\Delta^*\omega_t)}{d\omega_t} \right|^2 \right\rangle dt < \infty.$$

Here a $\nu \in \mathcal{D}^*$ is absolutely continuous with respect to a $w \in \mathcal{M}_r\left(\mathbb{R}^d\right)$ if

$$\langle \nu, f \rangle = \langle w, hf \rangle, \ \forall f \in \mathcal{D},$$

for some locally w-integrable measurable function h on $\mathbb{R}^d$; and write $h = \frac{d\nu}{dw}$.

The rate function and conjecture

- For any $\omega \in H_\mu^\sigma$, let

$$g_t = g_t^\omega := \frac{d(\dot{\omega}_t - \sigma\Delta^*\omega_t)}{d\omega_t}, \; t \in [0,1].$$

Put

$$I_\mu^{\sigma,\rho}(\omega) = \begin{cases} \frac{1}{2\rho}\int_0^1 \left\langle \omega_t, |g_t|^2 \right\rangle dt, & \text{if } \omega \in H_\mu^\sigma, \\ \infty, & \text{if } \omega \in \Omega_\mu \setminus H_\mu^\sigma. \end{cases}$$

Note the introduction of H_μ^σ and $I_\mu^{\sigma,\rho}$ was attributed to
[K. Fleischmann, J. Gärtner, I. Kaj (1996) *Can. J. Math.*],
[K. Fleischmann, J. Gärtner, I. Kaj (1993) preprint].

The rate function and conjecture

- Assume $\mu \in \mathcal{M}_F\left(\mathbb{R}^d\right) \setminus \{0\}$. Let

$$H^{\sigma}_{\mu,F} = H^{\sigma}_{\mu} \cap \Omega_{\mu,F};$$

and for any $\omega \in \Omega_{\mu,F}$, put

$$I^{\sigma,\rho}_{\mu,F}(\omega) = \infty I_{\left[\omega \in \Omega_{\mu,F} \setminus H^{\sigma}_{\mu,F}\right]} + I^{\sigma,\rho}_{\mu}(\omega) I_{\left[\omega \in H^{\sigma}_{\mu,F}\right]}.$$

The rate function and conjecture

• The affirmative answer to the following conjecture has been expected for a long time (16 years).

Conjecture 1. For any $\mu \in \mathcal{M}_r\left(\mathbb{R}^d\right) \setminus \mathcal{M}_F\left(\mathbb{R}^d\right)$, on Ω_μ, $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ satisfies a LDP with good rate function $I_\mu^{\sigma,\rho}$ as $\epsilon \downarrow 0$. Whereas for any $\mu \in \mathcal{M}_F\left(\mathbb{R}^d\right) \setminus \{0\}$, on $\Omega_{\mu,F}$, $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ satisfies a LDP with good rate function $I_{\mu,F}^{\sigma,\rho}$ as $\epsilon \downarrow 0$.

• Notice the above important and deep conjecture was (implicitly) stated in

[K. Fleischmann, J. Gärtner, I. Kaj (1996) *Can. J. Math.*].

It even has been unknown whether the conjecture holds for a certain initial measure.

Outline

Historical backgrounds on the conjecture

- Assume $\mu \neq 0$. In [A. Schied (1996) *PTRF*], LDP for $\left\{P_{\mu}^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ as $\epsilon \downarrow 0$ was given and the good rate function was represented by a variation formula, which was expected to be $I_{\mu}^{\sigma,\rho}$. See also [A. Schied (1995)].

- We point out that though the two variation formulae in [A. Schied (1996) *PTRF*] and [K. Fleischmann, J. Gärtner, I. Kaj (1996) *Can. J. Math.*] representing the good rate functions for the mentioned LDP respectively seem different, they are identical due to that the rate function for any LDP on a Polish space is unique ([A. Dembo, O. Zeitouni (1998) pp.117-118 Remarks]).

Historical backgrounds on the conjecture

• Recall along the lines of [K. Fleischmann, I. Kaj (1994) *Ann. Inst. H. Poincaré.*], a weak LDP for $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ as $\epsilon \downarrow 0$ on Ω_μ was established in a topology weaker than the compact-open topology by an original preprint of [K. Fleischmann, J. Gärtner, I. Kaj (1993)].

• Then [K. Fleischmann, J. Gärtner, I. Kaj (1996) *Can. J. Math.*] also derived LDP for $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ as $\epsilon \downarrow 0$. When replacing the underlying process, Brownian motion, by a killed or reflected Brownian motion in a bounded domain of $\mathbb{R}^d$, they got the corresponding explicit rate function $I_\mu^{\sigma,\rho}$. However, for the Brownian motion case, relying on an additional unproven 'Hypothesis of local blow-up', they proved the variation formula which represents the rate function to be $I_\mu^{\sigma,\rho}$.

Historical backgrounds on the conjecture

• [B. Djehiche, I. Kaj (1995) *Ann. Probab.*] took a Hamiltonian approach and a Girsanov transformation technique to prove a LDP result for some measure-valued jump processes, and compared the rate function therein with that in [K. Fleischmann, J. Gärtner, I. Kaj (1996) *Can. J. Math.*] of the LDP for $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ as $\epsilon \downarrow 0$, and hoped to extend their method to the LDP for $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$.

• Then [B. Djehiche, I. Kaj (1999) *Bull. Sci. Math.*] took the just mentioned method to prove directly upper and lower bounds of LDP for $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ as $\epsilon \downarrow 0$. In their paper, though the rate function for the upper bound LDP was $I_\mu^{\sigma,\rho}$, it required a restriction to more regular paths to get a lower large deviation bound.

Historical backgrounds on the conjecture

• Note the variation for the considered LDP is not covered by the standard theory of variational methods (e.g., [M. Struwe (2000)]); and [K. Fleischmann, J. Gärtner, I. Kaj (1996) *Can. J. Math.*] proposed a local blow-up hypothesis to calculate the mentioned variation, however there is no statement on local blow-up properties in the PDE theory implying this hypothesis.

• The mentioned LDP for finite initial measures has some difference from that for infinite initial measures was not discovered until 2009. (This difference is stated explicitly in Conjecture 1.)

Historical backgrounds on the conjecture

• **Thus, no satisfactory derivation of the full LDP for** $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}$ **as** $\epsilon \downarrow 0$ **exists to date.**

Outline

Main Theorems

- **Thanks to previous results on the LDP for SBMs, and the Le Gall's Brownian snake (e.g., [J. F. Le Gall (1999)]), we make an important breakthrough on Conjecture 1 in May 2009 (the related proof is highly nontrivial).**

Main Theorems

- **Theorem 2 [May, 2009].** (**i**) Given any $\mu \in \mathcal{M}_F\left(\mathbb{R}^d\right) \setminus \{0\}$. On $\Omega_{\mu,F}$, $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ satisfies a LDP with the good rate function $I_{\mu,F}^{\sigma,\rho}$ as $\epsilon \downarrow 0$; namely, for any closed subset $C \subseteq \Omega_{\mu,F}$,

$$\limsup_{\epsilon\downarrow 0} \epsilon \log P_\mu^{\sigma,\epsilon\rho}\left[\omega \in C\right] \leq -\inf_{\omega\in C} I_{\mu,F}^{\sigma,\rho}(\omega),$$

and for any open subset $O \subseteq \Omega_{\mu,F}$,

$$\liminf_{\epsilon\downarrow 0} \epsilon \log P_\mu^{\sigma,\epsilon\rho}\left[\omega \in O\right] \geq -\inf_{\omega\in O} I_{\mu,F}^{\sigma,\rho}(\omega),$$

and $I_{\mu,F}^{\sigma,\rho}$ is lower semicontinuous, $\left\{\omega \in \Omega_{\mu,F} \;\middle|\; I_{\mu,F}^{\sigma,\rho}(\omega) \leq a\right\}$ is compact in $\Omega_{\mu,F}$ for any $a \in [0,\infty)$. Furthermore, there is no $\omega \in H_\mu^\sigma$ with $\int_0^1 \omega_t\left(\mathbb{R}^d\right)\, dt = \infty$, and hence $H_\mu^\sigma = H_{\mu,F}^\sigma$.

Main Theorems

- **Theorem 2 [May, 2009].** (**ii**) Assume $\frac{1}{2} < r \leq 1$ and $d = 1$. For any $\mu \in \mathcal{M}_r\left(\mathbb{R}^1\right) \setminus \mathcal{M}_F\left(\mathbb{R}^1\right)$, on Ω_μ, $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ satisfies a LDP with the good rate function $I_\mu^{\sigma,\rho}$ as $\epsilon \downarrow 0$. That is, for any closed subset $C \subseteq \Omega_\mu$,

$$\limsup_{\epsilon\downarrow 0} \epsilon \log P_\mu^{\sigma,\epsilon\rho}\left[\omega \in C\right] \leq -\inf_{\omega\in C} I_\mu^{\sigma,\rho}(\omega);$$

and for any open subset $O \subseteq \Omega_\mu$,

$$\liminf_{\epsilon\downarrow 0} \epsilon \log P_\mu^{\sigma,\epsilon\rho}\left[\omega \in O\right] \geq -\inf_{\omega\in O} I_\mu^{\sigma,\rho}(\omega);$$

and $I_\mu^{\sigma,\rho}$ is lower semicontinuous, $\left\{\omega \in \Omega_\mu \;\middle|\; I_\mu^{\sigma,\rho}(\omega) \leq a\right\}$ is compact in Ω_μ for any $a \in [0,\infty)$.

Main Theorems

- It is rather surprising that

$$H_{\mu}^{\sigma} = H_{\mu,F}^{\sigma} \text{ for any } \mu \in \mathcal{M}_F\left(\mathbb{R}^d\right) \setminus \{0\}.$$

- The fact we have to resort to certain restrictions (finite initial measures or a restricted class of infinite initial measures for the case $d = 1$) points to the subtleness and difficulty of Conjecture 1. **Fortunately, an ingenious and short proof is discovered to prove the conjecture holds for all infinite initial measures in June 2010.**

Main Theorems

- **Theorem 3 [June, 2010].** For any $\mu \in \mathcal{M}_r\left(\mathbb{R}^d\right) \setminus \mathcal{M}_F\left(\mathbb{R}^d\right)$, on Ω_μ, $\left\{P_\mu^{\sigma,\epsilon\rho}\right\}_{\epsilon>0}$ satisfies a LDP with the good rate function $I_\mu^{\sigma,\rho}$ as $\epsilon \downarrow 0$. That is, for any closed subset $C \subseteq \Omega_\mu$,

$$\limsup_{\epsilon\downarrow 0} \epsilon \log P_\mu^{\sigma,\epsilon\rho}\left[\omega \in C\right] \leq -\inf_{\omega\in C} I_\mu^{\sigma,\rho}(\omega);$$

and for any open subset $O \subseteq \Omega_\mu$,

$$\liminf_{\epsilon\downarrow 0} \epsilon \log P_\mu^{\sigma,\epsilon\rho}\left[\omega \in O\right] \geq -\inf_{\omega\in O} I_\mu^{\sigma,\rho}(\omega);$$

and $I_\mu^{\sigma,\rho}$ is lower semicontinuous, $\left\{\omega \in \Omega_\mu \;\middle|\; I_\mu^{\sigma,\rho}(\omega) \leq a\right\}$ is compact in Ω_μ for any $a \in [0,\infty)$.

Main Theorems

• **So far we have concluded the long-time attacking on the mentioned conjecture.**

Outline

Question

• For $d=1$, SBM $(X_t)_{t\in(0,1]}$ under $P_\mu^{\sigma,\epsilon\rho}$ has a continuous density function $\widetilde{X}_t(x)$ in $(t,x)\in(0,1]\times\mathbb{R}^1$ with respect to the Lebesgue measure dx solving the following SPDE in the mild sense:

$$d\widetilde{X}_t=\sigma\Delta\widetilde{X}_t\ dt+\sqrt{\epsilon\rho\widetilde{X}_t}\ dW_t,\ \ t\in(0,1].$$

where W_t is a space-time white noise.

• We conjecture that on some suitable continuous function space E on $\mathbb{R}^1$ endowed with a suitable topology so that E can be a Polish space, there is an explicit large deviation theorem for density function processes $\left(\widetilde{X}_t(\cdot)\right)_{t\in[0,1]}$ under $P_\mu^{\sigma,\epsilon\rho}$ on related continuous trajectory space $C([0,1],E)$.

Question

- To prove the just mentioned conjecture, it suffices to check the related exponential tightness on $C([0,1],E)$. In this case, the existing approach does not work (due to that the unbounded diffusion coefficient in the mentioned SPDE is not locally Lipschitzian and it is unknown the SPDE has the unique strong solution).

The End

- The talk is based on the following paper:

  Xiang Kai-Nan. (2010). An Explicit Schilder type theorem for super-Brownian motions (51 pages). *Comm. Pure. Appl. Math.* (To appear).

and a preprint:

An explicit Schilder type theorem for super-Brownian motions: infinite initial measures. Preprint, 2010.

Thank You !